Characterization of order structures avoiding three-term arithmetic progressions

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The 2026 Thammasat-Kyushu Symposium
on Mathematics and Statistics

มหาวิทยาลัยธรรมศาสตร์

Introduction

Brief self-introduction

As a PhD student, I studied **classical real analysis**. More specifically, I studied how badly "most" continuous functions behave.
As a postdoc, I collaborated with a non-life insurance company.
Nowadays I mainly study **multiple zeta values** (a generalization of the Riemann zeta values):

$$\zeta(k_1, \ldots, k_r) := \sum_{1 \le n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \ldots n_r^{k_r}}.$$

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Today’s talk

This talk is based on the following paper:

> Minoru Hirose and S. S.,
> *Characterization of order structures avoiding three-term arithmetic progressions*,
> Order **42** (2025), 231–239,
> `https://doi.org/10.1007/s11083-024-09677-7` (open access).

Puzzle

Take a permutation of $\{0, 1, 2, 3, 4\}$ (there are $5! = 120$ permutations).

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Does it contain a three-term arithmetic progression as a subsequence?
If the permutation is $4, 1, 2, 3, 0$, then the answer is **yes** because it contains $1, 2, 3$.
If the permutation is $3, 4, 1, 2, 0$, then the answer is **yes** because it contains $4, 2, 0$ as a **subsequence**.

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Does there exist a permutation for which the answer is **no**?

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Does there exist a permutation for which the answer is **no**?
If the permutation is $0, 4, 2, 1, 3$, then the answer is **no** because it does not contain $0, 1, 2$; $1, 2, 3$; $2, 3, 4$; $0, 2, 4$; or their reversals as a subsequence.

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Take a permutation of $\{0, 1, 2, 3, 4\}$ (there are $5! = 120$ permutations).
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Problem

Given a nonnegative integer n, does there exist a permutation of $\{0, \ldots, n\}$ that does not contain a three-term AP as a subsequence?
(AP = arithmetic progression)

Answer: Yes!

Proposition

For every nonnegative integer n, there exists a permutation of $\{0, \ldots, n\}$ that does not contain a three-term AP as a subsequence.

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Does there exist a permutation of $\mathbb{N} = \{0, 1, \ldots\}$ that does not contain a three-term AP as a subsequence?

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Problem

Does there exist a permutation of $\mathbb{N} = \{0, 1, \dots\}$ that does not contain a three-term AP as a subsequence?

Theorem (Davis, Entringer, Graham, and Simmons, 1977)

No such permutation exists.

Observation

Recall the construction for finite n:

even				odd				
$\equiv 0$		$\equiv 2$		$\equiv 1$		$\equiv 3$		$\pmod 4$
$\equiv 0$	$\equiv 4$	$\equiv 2$	$\equiv 6$	$\equiv 1$	$\equiv 5$	$\equiv 3$	$\equiv 7$	$\pmod 8$

$$\vdots$$

This gives a total order $\leq'$ on $\mathbb{N}$.

The total order $\leq'$ has the property that there is no AP a, b, c with $a <' b <' c$.

Order structure of $\leq'$

The order $\leq'$ can be illustrated as follows:

$0 \qquad\qquad\qquad\qquad 1$

$2 \qquad\qquad\qquad\qquad 3$

$4 \qquad 6 \qquad 5 \qquad 7$

$8 \quad 12 \quad 10 \quad 14 \quad 9 \quad 13 \quad 11 \quad 15$

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$$
\begin{array}{ccccccccccccccccc}
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 & & & & 2 & & & & & & & & 3 & & & & \\
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\end{array}
$$

This has the same structure as the **Stern-Brocot tree**:

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\begin{array}{ccccccccccccccccc}
\frac{0}{1} & & & & & & & & \frac{1}{1} & & & & & & & & \left(\frac{1}{0}\right) \\
 & & & & \frac{1}{2} & & & & & & & & \frac{2}{1} & & & & \\
 & & \frac{1}{3} & & & & \frac{2}{3} & & & & \frac{3}{2} & & & & \frac{3}{1} & & \\
 & \frac{1}{4} & & \frac{2}{5} & & \frac{3}{5} & & \frac{3}{4} & & \frac{4}{3} & & \frac{5}{3} & & \frac{5}{2} & & \frac{4}{1} &
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Every nonnegative rational number appears exactly once, in the usual order.

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Every nonnegative rational number appears exactly once, in the usual order.
It follows that $(\mathbb{N}, \leq')$ and $(\mathbb{Q}_{\geq 0}, \leq)$ are isomorphic as ordered sets.

Main theorem

We have constructed a total order $\leq'$ on $\mathbb{N}$ such that

- there is no AP a, b, c with $a <' b <' c$;
- $(\mathbb{N}, \leq') \cong (\mathbb{Q}_{\geq 0}, \leq)$.

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In other words, there exists a bijection $f\colon \mathbb{N} \to \mathbb{Q}_{\geq 0}$ such that there is no AP a, b, c with $f(a) < f(b) < f(c)$.

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On the other hand, the theorem of Davis, Entringer, Graham, and Simmons implies that there does not exist a bijection $f \colon \mathbb{N} \to \mathbb{N}$ such that there is no AP a, b, c with $f(a) < f(b) < f(c)$.

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Problem

Given a totally ordered set $(X, \preceq)$, determine whether there exists a bijection $f\colon \mathbb{N} \to X$ such that there is no AP a, b, c with $f(a) \prec f(b) \prec f(c)$.

Main theorem

Theorem (Hirose and S.)

Let $(X, \preceq)$ be a countably infinite totally ordered set. Then the following are equivalent:

(1) There exists a bijection $f\colon \mathbb{N} \to X$ such that there is no AP a, b, c with $f(a) \prec f(b) \prec f(c)$.

(2) The topological space X endowed with the order topology has no isolated points, namely there is no $p \in X$ such that either
- $\{x \in X \mid x \prec x_0\} = \{p\}$ for some $x_0 \in X$,
- $\{x \in X \mid x \succ x_0\} = \{p\}$ for some $x_0 \in X$, or
- $\{x \in X \mid x_0 \prec x \prec x_1\} = \{p\}$ for some $x_0, x_1 \in X$.

Remark

With $\mathbb{N}$ replaced by $\mathbb{Z}$, the theorem remains valid with (2) unchanged. With $\mathbb{N}$ replaced by $\mathbb{Q}$, the theorem remains valid with (2) strengthened by further requiring that X have no maximum or no minimum.