Characterization of order structures avoiding three-term arithmetic progressions

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Introduction

Brief self-introduction

As a PhD student, I studied **classical real analysis**. More specifically, I studied the differentiability properties of typical continuous functions.

As a postdoc, I collaborated with a non-life insurance company.

Nowadays I mainly study **multiple zeta values** (a generalization of the Riemann zeta values):

$$\zeta(k_1, \dots, k_r) := \sum_{1 \le n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \dots n_r^{k_r}}.$$

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Today's talk

This talk is based on the following paper:

Minoru Hirose and S. S.,
Characterization of order structures avoiding three-term arithmetic progressions,
Order **42** (2025), 231–239,
`https://doi.org/10.1007/s11083-024-09677-7` (open access).

Puzzle

Problem

Does there exist a permutation $a_0, \ldots, a_{2026}$ of $\{0, \ldots, 2026\}$ that contains no three-term arithmetic progression a_i, a_j, a_k with $0 \leq i < j < k \leq 2026$?

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History

This problem (with 2026 replaced by n) was proposed by R. C. Entringer and D. E. Jackson in the Elementary Problems section of the American Mathematical Monthly in 1973.

Solutions were provided by several people, including Tom Odda ($=$ R. L. Graham), R. C. Lyndon, and H. E. Thomas, Jr.

In what follows, we write AP to mean arithmetic progression.

Answer: Yes!

Proposition

For every nonnegative integer n, there exists a permutation $a_0, \ldots, a_n$ of $\{0, \ldots, n\}$ that contains no AP a_i, a_j, a_k with $0 \leq i < j < k \leq n$.

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$\vdots$

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Proof that this construction works

Suppose that a_i, a_j, a_k is an AP. Then a_i and a_k are both even or both odd because $a_i + a_k = 2a_j \equiv 0 \pmod 2$. It follows that a_i, a_j, a_k are in the same block modulo 2.

If a_i, a_j, a_k are all even, then $a_i + a_k = 2a_j \equiv 0 \pmod 4$.

If a_i, a_j, a_k are all odd, then $a_i + a_k = 2a_j \equiv 2 \pmod 4$.

Hence a_i, a_j, a_k are in the same block modulo 4. Continue this argument.

Infinite permutation

Does there exist a permutation $a_0, a_1, \dots$ of $\mathbb{N} = \{0, 1, \dots\}$ that contains no AP a_i, a_j, a_k with $0 \le i < j < k$?

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Theorem (Davis, Entringer, Graham, and Simmons, 1977)

No such permutation exists.

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Proof

Let $a_0, a_1, \ldots$ be any permutation.
We shall find j and k such that $0 < j < k$ and a_0, a_j, a_k is an AP.
Let j be the smallest such that $a_j > a_0$.
Choose k for which $a_k = 2a_j - a_0$.
We have $k > j$ because

- $a_k > a_0$ and so $k \geq j$ by the choice of j;
- $a_k > a_j$ and so $k \neq j$.

Observation

Recall the construction for finite n:

<table>
<tr><td colspan="4">even</td><td colspan="4">odd</td><td></td></tr>
<tr><td colspan="2">$\equiv 0$</td><td colspan="2">$\equiv 2$</td><td colspan="2">$\equiv 1$</td><td colspan="2">$\equiv 3$</td><td>$\pmod 4$</td></tr>
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</table>

$$\vdots$$

This gives a total order $\leq'$ on $\mathbb{N}$.

The total order $\leq'$ has the property that there is no AP a, b, c with $a <' b <' c$.

Order structure of $\leq'$

The order $\leq'$ can be illustrated as follows:

$0 \qquad 1$

$2 \qquad 3$

$4 \qquad 6 \qquad 5 \qquad 7$

$8 \qquad 12 \qquad 10 \qquad 14 \qquad 9 \qquad 13 \qquad 11 \qquad 15$

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$$\begin{array}{ccccccccccccccccc}
0 & & & & & & & & 1 & & & & & & & & \\
 & & & & 2 & & & & & & & & 3 & & & & \\
 & & 4 & & & & 6 & & & & 5 & & & & 7 & & \\
 & 8 & & 12 & & 10 & & 14 & & 9 & & 13 & & 11 & & 15 &
\end{array}$$

This has the same structure as the **Stern-Brocot tree**:

$$\begin{array}{ccccccccccccccccc}
\frac{0}{1} & & & & & & & & \frac{1}{1} & & & & & & & & \left(\frac{1}{0}\right) \\
 & & & & \frac{1}{2} & & & & & & & & \frac{2}{1} & & & & \\
 & & \frac{1}{3} & & & & \frac{2}{3} & & & & \frac{3}{2} & & & & \frac{3}{1} & & \\
 & \frac{1}{4} & & \frac{2}{5} & & \frac{3}{5} & & \frac{3}{4} & & \frac{4}{3} & & \frac{5}{3} & & \frac{5}{2} & & \frac{4}{1} &
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Every nonnegative rational number appears exactly once, in the usual order.

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Every nonnegative rational number appears exactly once, in the usual order.
It follows that $(\mathbb{N}, \leq')$ and $(\mathbb{Q}_{\geq 0}, \leq)$ are isomorphic as ordered sets.

Stern-Brocot tree and continued fractions

Stern-Brocot tree:

$\frac{0}{1}$ $\qquad\qquad\qquad\qquad$ $\frac{1}{1}$ $\qquad\qquad\qquad\qquad$ $\left(\frac{1}{0}\right)$

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$\frac{1}{3}$ $\quad$ $\frac{2}{3}$ $\quad$ $\frac{3}{2}$ $\quad$ $\frac{3}{1}$

$\frac{1}{4}$ $\frac{2}{5}$ $\frac{3}{5}$ $\frac{3}{4}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{5}{2}$ $\frac{4}{1}$

Using the notation $[a; b, c] = a + \frac{1}{b+\frac{1}{c}}$, the tree can be written as follows:

$[0]$ $\qquad\qquad\qquad\qquad$ $[1]$

$[0;2]$ $\qquad\qquad$ $[2]$

$[0;3]$ $\quad$ $[0;1,2]$ $\quad$ $[1;2]$ $\quad$ $[3]$

$[0;4]$ $[0;2,2]$ $[0;1,1,2]$ $[0;1,3]$ $[1;3]$ $[1;1,2]$ $[2;2]$ $[4]$

Each node $[a_0; a_1, \dots, a_n]$ with $a_n \geq 2$ has two children $[a_0; a_1, \dots, a_{n-1}, a_n + 1]$ and $[a_0; a_1, \dots, a_{n-1}, a_n - 1, 2]$.

Main theorem

We have constructed a total order $\leq'$ on $\mathbb{N}$ such that

- there is no AP a, b, c with $a <' b <' c$;
- $(\mathbb{N}, \leq') \cong (\mathbb{Q}_{\geq 0}, \leq)$.

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In other words, there exists a bijection $f\colon \mathbb{N} \to \mathbb{Q}_{\geq 0}$ such that there is no AP a, b, c with $f(a) < f(b) < f(c)$.

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On the other hand, the theorem of Davis, Entringer, Graham, and Simmons implies that there does not exist a bijection $f \colon \mathbb{N} \to \mathbb{N}$ such that there is no AP a, b, c with $f(a) < f(b) < f(c)$.

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Problem

Given a totally ordered set $(X, \preceq)$, determine whether there exists a bijection $f\colon \mathbb{N} \to X$ such that there is no AP a, b, c with $f(a) \prec f(b) \prec f(c)$.

Main theorem

Theorem (Hirose and S.)

Let $(X, \preceq)$ be a countably infinite totally ordered set. Then the following are equivalent:

(1) There exists a bijection $f\colon \mathbb{N} \to X$ such that there is no AP a, b, c with $f(a) \prec f(b) \prec f(c)$.

(2) The topological space X endowed with the order topology has no isolated points, namely there is no $p \in X$ such that either
- $\{x \in X \mid x \prec x_0\} = \{p\}$ for some $x_0 \in X$,
- $\{x \in X \mid x \succ x_0\} = \{p\}$ for some $x_0 \in X$, or
- $\{x \in X \mid x_0 \prec x \prec x_1\} = \{p\}$ for some $x_0, x_1 \in X$.

Remark

With $\mathbb{N}$ replaced by $\mathbb{Z}$, the theorem remains valid with (2) unchanged. With $\mathbb{N}$ replaced by $\mathbb{Q}$, the theorem remains valid with (2) strengthened by further requiring that X have no maximum or no minimum.