t-adic symmetric multiple zeta values for indices in which 1 and 3 appear alternately

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Introduction

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This talk is based on

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Indices in which 1 and 3 appear alternately

An **index** is a finite (possibly empty) sequence $\boldsymbol{k} = (k_1, \ldots, k_r)$ of positive integers.

In this talk, a **good** index means an index in which 1 and 3 appear alternately.
E.g. (3), $(1, 3, 1)$, and $(3, 1, 3, 1) = (\{3, 1\}^2)$ are good indices.
We also regard $\emptyset$ as a good index.

We classify good indices into the following four types:

- **Type 11**: $\boldsymbol{k}$ starts with 1 and ends with 1, i.e. $\boldsymbol{k} = (\{1, 3\}^n, 1)$.
- **Type 13**: $\boldsymbol{k}$ starts with 1 and ends with 3, i.e. $\boldsymbol{k} = (\{1, 3\}^n)$.
- **Type 31**: $\boldsymbol{k}$ starts with 3 and ends with 1, i.e. $\boldsymbol{k} = (\{3, 1\}^n)$.
- **Type 33**: $\boldsymbol{k}$ starts with 3 and ends with 3, i.e. $\boldsymbol{k} = (\{3, 1\}^n, 3)$.

Why are we interested in good indices?
$\longrightarrow$ Because the multiple zeta values for good indices are polynomials of the Riemann zeta values.

Multiple zeta values for good indices

For each index $\boldsymbol{k} = (k_1, \dots, k_r)$ that is admissible (i.e. $r = 0$ or $k_r \geq 2$), we define the **multiple zeta value** (MZV) by

$$\zeta(k_1, \dots, k_r) \coloneqq \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \in \mathbb{R}.$$

Type 13 (conjectured by Zagier (1994) and proved by Borwein, Bradley, Broadhurst, and Lisoněk (1998/2001)):

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!} \in \mathbb{Q}\pi^{4n}.$$

Type 33 (Bowman and Bradley (2003)):

$$\zeta(\{3,1\}^n, 3) = 4^{-n} \sum_{i=0}^{n} (-1)^i \zeta(4i+3) \zeta(\{4\}^{n-i})$$
$$\in \mathbb{Q}[\pi^2, \zeta(3), \zeta(5), \dots] \eqqcolon \mathrm{Rie}.$$

Remark: $\zeta(\{4\}^n) = 2^{2n+1}\pi^{4n}/(4n+2)! \in \mathbb{Q}\pi^{4n}$.

Regularized multiple zeta values for good indices

For non-admissible indices $\boldsymbol{k}$, we can consider

- the harmonic (stuffle) regularization $\zeta^{*}(\boldsymbol{k}) \in \mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\text{MZVs}\}$ and
- the shuffle regularization $\zeta^{ш}(\boldsymbol{k}) \in \mathcal{Z}$.

All non-admissible indices that appear in this talk have only one trailing 1, and so we may write ζ without distinguishing ζ^{*} and $\zeta^{ш}$.

Types 11 and 31 (Bachmann and Charlton (2020)):

$$\zeta(\{1,3\}^n,1) = 2^{-2n+1}\sum_{i=0}^{n}(-1)^i\zeta(4i+1)\zeta(\{4\}^{n-i}) \in \mathrm{Rie},$$

$$\zeta(\{3,1\}^n) = 2^{-2n+3}\sum_{\substack{1\le i\le n-1\\ 0\le j\le n-i-1}}(-1)^{i+j}\zeta(4i+1)\zeta(4j+3)\zeta(\{4\}^{n-i-j-1}) + (-1)^n\sum_{i=0}^{n}4^{-i}\zeta^{\star}(\{4\}^i)\zeta(\{4\}^{n-i}) \in \mathrm{Rie}.$$

t-adic symmetric MZVs for good indices

In summary, we have **$\zeta(\boldsymbol{k}) \in \mathrm{Rie}$ for all good indices $\boldsymbol{k}$**.

What about the t-adic symmetric multiple zeta values for good indices?

For each good index $\boldsymbol{k} = (k_1, \dots, k_r)$, we define the **$t$-adic symmetric multiple zeta value** (t-adic SMZV) by

$$\zeta_t(\boldsymbol{k}) := \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta(k_1, \dots, k_i) \sum_{l_{i+1},\dots,l_r \geq 0} \left(\prod_{j=i+1}^{r} \binom{k_j + l_j - 1}{l_j} \right) \times \zeta(k_r + l_r, \dots, k_{i+1} + l_{i+1}) t^{l_{i+1}+\cdots+l_r} \in \mathcal{Z}[[t]].$$

The constant term of the t-adic SMZV is

$$[t^0]\zeta_t(\boldsymbol{k}) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta(k_1, \dots, k_i) \zeta(k_r, \dots, k_{i+1}) \in \mathcal{Z}.$$

Kaneko-Zagier conjecture

The Kaneko-Zagier conjecture asserts that the **$\mathcal{A}$-MZVs**

$$\begin{aligned}\zeta_{\mathcal{A}}(k_1,\ldots,k_r) &:= \left(\sum_{1\le n_1<\cdots<n_r<p} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \bmod p\right)_p \\ &\in \mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z}\right),\end{aligned}$$

where p runs over all primes, and the **$\mathcal{S}$-MZVs**

$$\begin{aligned}\zeta_{\mathcal{S}}(k_1,\ldots,k_r) &= [t^0]\zeta_t(k_1,\ldots,k_r) \bmod \pi^2\mathcal{Z} \\ &\in \mathcal{Z}/\pi^2\mathcal{Z}\end{aligned}$$

satisfy the same relations.

Refined Kaneko-Zagier conjecture

The refined Kaneko-Zagier conjecture asserts that the **$\widehat{\mathcal{A}}$-MZVs**

$$\zeta_{\widehat{\mathcal{A}}}(k_1,\dots,k_r) := \left(\left(\sum_{1\le n_1<\cdots<n_r<p}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \bmod p^m\right)_p\right)_m$$
$$\in \widehat{\mathcal{A}} := \varprojlim_m \left(\left(\prod_p \mathbb{Z}/p^m\mathbb{Z}\right)\Big/\left(\bigoplus_p \mathbb{Z}/p^m\mathbb{Z}\right)\right)$$

and the **$\widehat{\mathcal{S}}$-MZVs**

$$\zeta_{\widehat{\mathcal{S}}}(\boldsymbol{k}) := \zeta_t(k_1,\dots,k_r) \bmod \pi^2\mathcal{Z}$$
$$\in (\mathcal{Z}/\pi^2\mathcal{Z})[[t]]$$

satisfy the same relations.

Constant term of t-SMZVs for good indices

Question

Is it true that $\zeta_t(\boldsymbol{k}) \in \mathrm{Rie}$ for all good indices $\boldsymbol{k}$? $\longrightarrow$ No in general.
$\longrightarrow$ For which $m \in \mathbb{Z}_{\geq 0}$ is it true that $[t^m]\zeta_t(\boldsymbol{k}) \in \mathrm{Rie}$ for good indices $\boldsymbol{k}$?

We begin by looking at $[t^0]\zeta_t(\boldsymbol{k})$.

If $\boldsymbol{k}$ is of type 11 or 33, then $[t^0]\zeta_t(\boldsymbol{k}) = 0 \in \mathrm{Rie}$, as illustrated by

$$[t^0]\zeta_t(1,3,1) = \zeta(1,3,1) - \zeta(1,3)\zeta(1) + \zeta(1)\zeta(1,3) - \zeta(1,3,1) = 0.$$

If $\boldsymbol{k}$ is of type 13 or 31, then the harmonic relation implies that

$$[t^0]\zeta_t(\{1,3\}^n) = [t^0]\zeta_t(\{3,1\}^n) = (-1)^n\zeta(\{4\}^n) \left(= \frac{2(-4)^n}{(4n+2)!}\pi^{4n} \right) \in \mathrm{Rie},$$

as illustrated by

$$\begin{aligned} [t^0]\zeta_t(1,3) &= \zeta(1,3) - \zeta(1)\zeta(3) + \zeta(3,1) \\ &= \zeta(1,3) - (\zeta(1,3) + \zeta(3,1) + \zeta(4)) + \zeta(3,1) \\ &= -\zeta(4). \end{aligned}$$

Coefficient of t in t-SMZVs for good indices

If $\boldsymbol{k}$ is a good index of type 11, 13, or 31, then $[t]\zeta_t(\boldsymbol{k}) \in \mathrm{Rie}$:

Theorem (Types 11, 13, and 31)

$$[t]\zeta_t(\{1,3\}^n,1) = \frac{(-4)^{n+1}}{(4n+4)!}\pi^{4n+2},$$

$$[t]\zeta_t(\{1,3\}^n) = \sum_{\substack{n_0,n_1\geq 0\\ n_0+n_1=n}} \frac{(-4)^{n_0+1}(2-(-4)^{-n_1})}{(4n_0+2)!}\pi^{4n_0}\zeta(4n_1+1)$$

$$-(-1)^n \sum_{\substack{n_0,n_1\geq 0\\ n_0+n_1=2n\\ n_0,n_1 \text{ odd}}} \frac{2^{n_0-n_1+2}}{(2n_0+2)!}\pi^{2n_0}\zeta(2n_1+1),$$

$$[t]\zeta_t(\{3,1\}^n) = (-1)^{n+1} \sum_{\substack{n_0,n_1\geq 0\\ n_0+n_1=2n}} \frac{(-1)^{n_0}2^{n_0-n_1+2}}{(2n_0+2)!}\pi^{2n_0}\zeta(2n_1+1).$$

In contrast, $[t]\zeta_t(3,1,3) = \frac{19\pi^8}{37800} + \frac{\pi^2}{6}\zeta(3)^2 - 5\zeta(3)\zeta(5) - \boldsymbol{\zeta(3,5)}$ is presumably not even in $\mathrm{Rie} + \pi^2\mathcal{Z}$.

Coefficient of t^2 for good indices of types 11 and 31

If $\boldsymbol{k}$ is a good index of type 11 or 31, then $[t^2]\zeta_t(\boldsymbol{k}) \in \mathrm{Rie}$:

Theorem (Types 11 and 31)

$$[t^2]\zeta_t(\{1,3\}^n,1) = (-1)^n \sum_{\substack{n_0,n_1\geq 0\\ n_0+n_1=2n+1}} \frac{(-1)^{n_1}2^{n_0-n_1+2}}{(2n_0+2)!}\pi^{2n_0}\zeta(2n_1+1),$$

$$[t^2]\zeta_t(\{3,1\}^n)$$

$$= (-1)^n \sum_{\substack{n_0,n_1,n_2\geq 0\\ n_0+n_1+n_2=2n}} \frac{(-1)^{n_0}2^{n_0-n_1-n_2+2}}{(2n_0+2)!}\pi^{2n_0}\zeta(2n_1+1)\zeta(2n_2+1).$$

Coefficient of t^2 for good indices of type 13

In contrast,

$$[t^2]\zeta_t(1,3,1,3)$$
$$=\frac{81}{8}\zeta(5)^2-\frac{103\pi^{10}}{935550}-\frac{\pi^4}{180}\zeta(3)^2-\frac{1}{4}\zeta(3)\zeta(7)+\frac{\pi^2}{12}\zeta(3)\zeta(5)+\frac{\pi^2}{6}\boldsymbol{\zeta(3,5)}$$

is presumably not in Rie. But it is in $\mathrm{Rie}+\pi^2\mathcal{Z}$.

Theorem (Kento Fujita (2025+), conjectured by Hirose, Murahara, S.)

$$[t^2]\zeta_t(\{1,3\}^n)$$
$$\equiv 2\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=n}}((-4)^{-n_1}-2)((-4)^{-n_2}-2)\zeta(4n_1+1)\zeta(4n_2+1)$$
$$-2(-4)^{-n}\sum_{\substack{n_1,n_2\geq 0\\ n_1+n_2=n-1}}\zeta(4n_1+3)\zeta(4n_2+3)\pmod{\pi^2\mathcal{Z}}.$$

Coefficient of t^3 for good indices of types 11 and 31

$$[t^3]\zeta_t(1,3,1) = -\frac{43\pi^8}{113400} - \frac{\pi^2}{12}\zeta(3)^2 + \frac{9}{2}\zeta(3)\zeta(5) + \boldsymbol{\zeta(3,5)},$$

$$\begin{aligned}[t^3]\zeta_t(3,1,3,1) = &-\frac{\pi^8}{1134}\zeta(3) - \frac{7\pi^4}{180}\zeta(7) - \frac{\pi^2}{12}\zeta(3)^3 + \frac{19}{4}\zeta(3)^2\zeta(5) \\ &+ 2\zeta(3)\boldsymbol{\zeta(3,5)} + \frac{\pi^6}{189}\zeta(5) - 15\pi^2\zeta(9) - 2\boldsymbol{\zeta(3,3,5)} \\ &+ \frac{605}{4}\zeta(11).\end{aligned}$$

These are presumably not in $\mathrm{Rie} + \pi^2\mathcal{Z}$.

First step towards a proof

For a good index $\boldsymbol{k}$, we want to compute the coefficients of t and t^2 in

$$\zeta_t(\boldsymbol{k}) = \sum_{i=0}^{r} (-1)^{k_{i+1}+\cdots+k_r} \zeta(k_1,\ldots,k_i) \sum_{l_{i+1},\ldots,l_r \geq 0} \left(\prod_{j=i+1}^{r} \binom{k_j + l_j - 1}{l_j} \right) \times \zeta(k_r + l_r, \ldots, k_{i+1} + l_{i+1}) t^{l_{i+1}+\cdots+l_r} \in \mathcal{Z}[[t]].$$

Since $\boldsymbol{k}$ is good, so are $(k_1,\ldots,k_i)$ and $(k_r,\ldots,k_{i+1})$.
Therefore we know the values of $\zeta(k_1,\ldots,k_i)$.
We shall evaluate

$$\sum_{\substack{l_{i+1},\ldots,l_r \geq 0 \\ l_{i+1}+\cdots+l_r = m}} \left(\prod_{j=i+1}^{r} \binom{k_j + l_j - 1}{l_j} \right) \zeta(k_r + l_r, \ldots, k_{i+1} + l_{i+1})$$

by using the shuffle product.

Evaluation using the shuffle product

For example, if $m = 1$ and $(k_r, \ldots, k_{i+1}) = (1, 3, 1, 3)$, then

$$\sum_{\substack{l_1,\ldots,l_4 \geq 0 \\ l_1+\cdots+l_4=1}} \binom{1+l_1-1}{l_1}\binom{3+l_2-1}{l_2}\binom{1+l_3-1}{l_3}\binom{3+l_4-1}{l_4}$$

$$\times \zeta(1+l_1, 3+l_2, 1+l_3, 3+l_4)$$

$$= \zeta(2,3,1,3) + 3\zeta(1,4,1,3) + \zeta(1,3,2,3) + 3\zeta(1,3,1,4)$$

$$= Z^{ш}(yxyx^2y^2x^2 + 3y^2x^3y^2x^2 + y^2x^2yxyx^2 + 3y^2x^2y^2x^3)$$

$$= Z^{ш}(x \,ш\, y^2x^2y^2x^2 - xy^2x^2y^2x^2)$$

$$= -Z^{ш}(xy^2x^2y^2x^2) \quad (Z^{ш}(x) = 0)$$

$$= -Z^{ш}(y^2x^2y^2x^2y) \quad \text{(duality)}$$

$$= -\zeta(1,3,1,3,1).$$